

Homework 6, due 10/7

1. Compute the integral

$$\int_0^{2\pi} \frac{d\theta}{3 - 2 \cos \theta}.$$

2. Compute the integral

$$\int_{-\infty}^{\infty} \frac{e^{x/2}}{\cosh x} dx.$$

Hint: consider the integral around a rectangle with vertices $\pm L$, $\pm L + i\pi$ as $L \rightarrow \infty$.

3. (a) Find the poles and corresponding residues of the meromorphic function $f(z) = \csc(\pi z)$.

- (b) Show that

$$\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^2} = \frac{\pi^2}{12}.$$

4. Let $H = \{z : \text{Im}(z) > 0\}$ denote the upper half plane. Find all biholomorphisms $f : H \rightarrow H$.

5. Let $f : D \rightarrow D$ be holomorphic on the unit disk D , such that $f(0) = 0$.

- (a) Prove that

$$|f(z) + f(-z)| \leq 2|z|^2$$

for all $z \in D$.

- (b) Suppose that $|f(z_0) + f(-z_0)| = 2|z_0|^2$ for some $z_0 \neq 0$. Show that then $f(z) = e^{i\theta} z^2$ for some constant $\theta \in \mathbf{R}$, for all $z \in D$.